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On the general solution for the modified Emden-type equation $\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0$

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Abstract

In this paper, we demonstrate that the modified equation of Emden type (MEE), $\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0$, is integrable either explicitly or by quadrature for any value of α and β . We also prove that the MEE possesses appropriate time-independent Hamiltonian functions for the full range of parameters α and β . In addition we show that the MEE is intimately connected with two well-known nonlinear models, namely the equation of force-free Duffing-type oscillator and the two-dimensional Lotka–Volterra equation, and thus the complete integrability of the latter two models can also be understood in terms of the MEE.

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1. Introduction

One of the well-discussed models in nonlinear dynamics is the modified equation of Emden type (MEE), also called the modified Painlevé–Ince equation,

$$\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0, \quad (1)$$

where the overdot denotes differentiation with respect to time and α and β are arbitrary parameters. This equation has received attention from both mathematicians and physicists for more than a century [1–5]. For example, in the nineteenth century Painlevé had studied this equation and identified general solutions for two parametric choices, namely (i) $\beta = \alpha^2/9$ and (ii) $\beta = -\alpha^2$ [1, 2]. The above differential equation (1) arises in a variety of mathematical problems such as univalued functions defined by second-order differential equations [6] and the Riccati equation [7]. On the other hand, physicists have shown that this equation arises in different contexts: for example, it occurs in the study of equilibrium configurations of a spherical gas cloud acting under the mutual attraction of its molecules and subject to the laws of thermodynamics [8, 9] and in the modelling of the fusion of pellets [10]. It also governs spherically symmetric expansion or collapse of a relativistically gravitating mass [11]. This equation can also be thought of as a one-dimensional analogue [7, 12] of the boson ‘gauge-theory’ equations introduced by Yang and Mills. Apart from the above, for the past

two decades or so the invariance and integrability properties of this equation alone have been studied in detail by a number of authors; see, for example, [13–23]. In a nutshell the MEE (1) has been found to possess an explicit general solution only for the following parametric choices, that is, (i) $\alpha = 0$, (ii) $\beta = 0$, (iii) $\beta = \alpha^2/9$ and (iv) $\beta = -\alpha^2$. While the cases (i) and (ii) can be integrated trivially, in the third case the equation is linearizable to a free particle equation and in the fourth case the general solution can be expressed in terms of the Weierstrass elliptic function [1–5, 13–24]. Equation (1) has also been noted to possess the Painlevé property only for certain values of $r = (\alpha/4\beta)(\alpha \pm \sqrt{\alpha^2 - 8\beta})$ [17, 19]. Finally we mention that equation (1) admits a two-parameter Lie point symmetry group for arbitrary values of α and β , while for the choice $\beta = \alpha^2/9$ equation (1) possesses eight-parameter Lie point symmetries [13]. However, the general solution for equation (1) with α and β arbitrary is yet to be explored.

Very recently [23], the present authors have studied a generalized version of this equation from a different perspective and shown that equation (1), for $\alpha^2 \geq 8\beta$, possesses a time-independent integral and admits a Hamiltonian formalism which in turn ensures its complete integrability [23]. However, due to the complicated form of the first integral the general solution was not obtained in the previous work. Keeping in mind the historical importance and popularity of this model, we kept exploring the general solution of equation (1) in phase space. Based on our investigation, in this paper we construct the time-independent integrals for equation (1) for arbitrary values of α and β (including the case $\alpha^2 < 8\beta$). Since the first integrals are not in a simple polynomial form, it is difficult to obtain the general solution by just directly integrating them. In order to overcome this difficulty, firstly we identify time-independent Hamiltonians from these time-independent integrals and making use of suitable canonical transformations we convert the Hamiltonians into standard forms. We then integrate the new Hamiltonians and obtain the general solutions or reduce to quadratures. In this way, we report the general solution for equation (1) for arbitrary values of α and β for the first time. Our motivation to explore this solution is, as we see below, also due to the fact that the MEE (1) is not a stand-alone model. It is intimately connected to an equation of force-free Duffing oscillator type (vide equation (31)) and two-dimensional Lotka–Volterra (LV) equation (vide equation (33)). The popularity and importance of these two models need no emphasis [23, 25–29]. Thus, exploring the general solution for equation (1) also serves to establish the complete integrability of these two models for appropriate parameters besides understanding the dynamics of the other models mentioned in the introduction.

The plan of the paper is as follows. In section 2 we give the time-independent integrals and corresponding Hamiltonians for the MEE (1). Using suitable canonical transformations, we obtain general solutions for the parameter ranges $\alpha^2 = 8\beta$, $\alpha^2 > 8\beta$ and $\alpha^2 < 8\beta$ separately in section 3. In section 4 we show that the MEE (1) is intimately connected with two other well-known nonlinear models, namely the equation of force-free Duffing oscillator type and two-dimensional Lotka–Volterra equation. Finally in section 5 we summarize our results.

2. Time-independent integrals and Hamiltonian description

As indicated in the introduction, the MEE (1) cannot be straightforwardly integrated. Making use of the modified Prele–Singer method developed by us recently [22, 23] following the earlier work of Duarte *et al* [30], in this section we firstly identify the first integrals separately for each of the three ranges (i) $\alpha^2 = 8\beta$, (ii) $\alpha^2 > 8\beta$ and (iii) $\alpha^2 < 8\beta$. Then we identify a suitable canonical Hamiltonian description for each of these cases.

2.1. Time-independent integrals

In [23] the time-independent integral for equation (1) with $\alpha^2 \geq 8\beta$ has been reported using the modified Prolle–Singer procedure. However, improving the ansatz given in [23] one can obtain the time-independent integrals for all values of α and β , and the method of deriving the integrals for equation (1) with α and β arbitrary is given in appendix A. Using this method, we identify the following time-independent first integrals for equation (1) with α and β arbitrary, that is,

Case 1. $\alpha^2 = 8\beta$ ($r = 2$)

$$I = \log(\alpha^2 x^2 + 4\alpha\dot{x}) - \frac{4\alpha\dot{x}}{\alpha^2 x^2 + 4\alpha\dot{x}}. \tag{2}$$

Case 2. $\alpha^2 > 8\beta$ ($r \neq 0, 1, 2$)

$$I = \frac{(r-1)}{(r-2)} \left(\dot{x} + \frac{(r-1)}{2r} \alpha x^2 \right)^{1-r} \left(\dot{x} + \frac{\alpha}{2r} x^2 \right). \tag{3}$$

Case 3. $\alpha^2 < 8\beta$

$$I = \frac{1}{2} \log(2\dot{x}^2 + \alpha x^2 \dot{x} + \beta x^4) + \frac{\alpha}{\omega} \tan^{-1} \left[\frac{\alpha\dot{x} + 2\beta x^2}{\omega\dot{x}} \right], \tag{4}$$

where

$$\omega = \sqrt{8\beta - \alpha^2} \quad \text{and} \quad r = \frac{\alpha}{4\beta} (\alpha \pm \sqrt{\alpha^2 - 8\beta}).$$

Note that $r = 0$ and 1 correspond to the trivial cases $\alpha = 0$ and $\beta = 0$, respectively, and so they are not considered here separately.

As it is very difficult to integrate equations (2)–(4) and obtain the general solutions by direct integration, we correlate these integrals with appropriate Hamiltonians. In the following, we briefly give the method of obtaining the Hamiltonian from the known time-independent integral.

2.2. Hamiltonian description

To explore the Hamiltonian description of (2), we assume the existence of a Hamiltonian

$$I(x, \dot{x}) = H(x, p) = p\dot{x} - L(x, \dot{x}), \tag{5}$$

where L is the Lagrangian and p is the canonically conjugate momentum. Then,

$$\frac{\partial I}{\partial \dot{x}} = \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}} \dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}} \dot{x}. \tag{6}$$

From (6) we identify

$$p = \int \frac{I_{\dot{x}}}{\dot{x}} d\dot{x} + f(x), \tag{7}$$

where $f(x)$ is an arbitrary function of x and the Lagrangian $L = p\dot{x} - I(x, \dot{x})$. Here, without loss of generality, we take $f(x) = 0$. Substituting integrals (2)–(4) into (7) and integrating the resultant integrals, we can obtain the expression for the canonical momentum p .

Substituting back the latter into equation (5) and simplifying the resultant equation, we arrive at the following Lagrangian:

$$L = \begin{cases} -\log\left(\dot{x} + \frac{\alpha}{4}x^2\right), & \alpha^2 = 8\beta \\ \frac{1}{(2-r)}\left(\dot{x} + \frac{(r-1)}{2r}\alpha x^2\right)^{(2-r)}, & \alpha^2 > 8\beta \\ \frac{1}{\omega}\left(\tan^{-1}\left[\frac{4\dot{x} + \alpha x^2}{\omega x^2}\right]\left(\frac{4\dot{x}}{x}\right) - \alpha \tan^{-1}\left[\frac{\alpha\dot{x} + 2\beta x^2}{\omega\dot{x}}\right]\right) \\ -\frac{1}{2}\log(2\dot{x}^2 + \alpha x^2\dot{x} + \beta x^4), & \alpha^2 < 8\beta \end{cases} \quad (8)$$

and Hamiltonian

$$H = \begin{cases} \log\left(-\frac{4\alpha}{p}\right) - \frac{\alpha}{4}px^2, & \alpha^2 = 8\beta \\ \frac{(r-1)}{(r-2)}(p)^{\frac{r-2}{r-1}} - \frac{(r-1)}{2r}\alpha x^2 p, & \alpha^2 > 8\beta \\ \frac{1}{2}\log\left[x^4 \sec^2\left(\frac{\omega}{4}x^2 p\right)\right] - \frac{\alpha}{4}x^2 p, & \alpha^2 < 8\beta \end{cases} \quad (9)$$

where the canonically conjugate momentum

$$p = \begin{cases} -\frac{1}{\left(\dot{x} + \frac{\alpha}{4}x^2\right)} & \alpha^2 = 8\beta \\ \left(\dot{x} + \frac{(r-1)}{2r}\alpha x^2\right)^{1-r} & \alpha^2 > 8\beta \\ \frac{4}{\omega x^2} \tan^{-1}\left[\frac{4\dot{x} + \alpha x^2}{2\omega x^2}\right], & \alpha^2 < 8\beta \end{cases} \quad (10)$$

respectively for the MEE (1).

One can easily check that the canonical equations of motion for the above Hamiltonians are nothing, but the equation of motion (1) of the MEE in the appropriate parametric regimes.

3. General solutions

In this section we consider each of the above three cases separately, namely (i) $\alpha^2 = 8\beta$, (ii) $\alpha^2 > 8\beta$ and (iii) $\alpha^2 < 8\beta$, and obtain their respective general solutions using suitable canonical transformations.

3.1. Case 1: $\alpha^2 = 8\beta$

To derive the general solution for this parametric choice, firstly we consider the Hamiltonian given in (9) for $\alpha^2 = 8\beta$ as

$$H = \log\left(-\frac{4\alpha}{p}\right) - \frac{\alpha}{4}px^2. \quad (11)$$

Introducing the canonical transformation

$$x = \frac{4P}{\alpha U}, \quad p = -\frac{\alpha U^2}{8}, \quad (12)$$

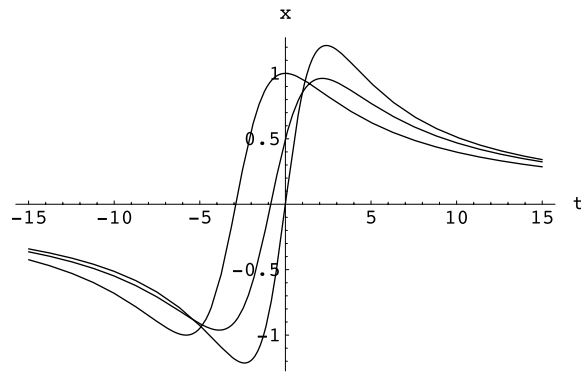


Figure 1. Solution plot for case $\alpha^2 = 8\beta$ for different initial conditions.

the Hamiltonian (11) can be recast into the standard form

$$H = \frac{1}{2}P^2 + \log\left(\frac{32}{U^2}\right) \equiv \hat{E}, \tag{13}$$

where \hat{E} is a constant. From the corresponding canonical equations $\dot{U} = P, \dot{P} = 2/U$, equation (13) can be rewritten as

$$E = \frac{1}{2}\dot{U}^2 - 2\log(U), \quad E = \hat{E} - \log(32). \tag{14}$$

Rewriting the above equation, we get

$$\frac{dU}{\sqrt{2E + 4\log(U)}} = dt. \tag{15}$$

Integrating equation (15), we obtain

$$U = \exp\left[-\frac{1}{2}(E + 2[\text{erf}^{-1}(z)]^2)\right], \tag{16}$$

where $z = (2(t_0 + it) \exp[E/2])/\sqrt{\pi}$ and t_0 is the second arbitrary constant of integration and erf is the error function [31].

Substituting equation (16) into equation (12) we get the general solution for equation (1), with the parametric choice $\beta = \alpha^2/8$, in the form (after some modifications)

$$x(t) = \frac{8}{\alpha i} \text{erf}^{-1}(z) \exp\left[\frac{1}{2}\left(E + 2[\text{erf}^{-1}(z)]^2\right)\right], \quad z = \frac{(2(t_0 + it) \exp[E/2])}{\sqrt{\pi}}. \tag{17}$$

In figure 1, we have plotted numerically the solution of MEE (1) for the parametric choice $\alpha^2 = 8\beta$ for different initial conditions.

3.2. Case 2: $\alpha^2 > 8\beta$

Now we consider the Hamiltonian for the case $\alpha^2 > 8\beta$ from (9), that is,

$$H = \frac{(r-1)}{(r-2)}(p)^{\frac{r-2}{r-1}} - \frac{(r-1)}{2r}\alpha x^2 p. \tag{18}$$

Interestingly, here we identify a canonical transformation for the Hamiltonian (18) in the form

$$x = -a\frac{P}{U}, \quad p = \frac{U^2}{2a}, \tag{19}$$

where $a = 2r/(\alpha(1-r))$, so that the Hamiltonian H in equation (18) can be rewritten as

$$H = \frac{1}{2}P^2 - kU^m \equiv E, \quad (20)$$

where

$$m = \frac{2(r-2)}{(r-1)} \quad \text{and} \quad k = \frac{(1-r)}{(2a)^{\frac{m}{2}}(r-2)} \quad \left(r = \frac{\alpha}{4\beta}(\alpha \pm \sqrt{\alpha^2 - 8\beta}) \right).$$

From the canonical equations, we then have $\dot{U} = P$ and $\dot{P} = mkU^{m-1}$. We also note here that the canonical transformation (12) for case 1 is a special case of (19) with the choice $a = -4/\alpha$, which implies that $r = 2$ or $\alpha^2 = 8\beta$ as it should be.

To obtain the general solution, we introduce a transformation $X = U^m$ in equation (20) so that the latter can be brought to a quadrature of the form

$$t - t_0 = \int \frac{X^{\frac{1-m}{m}} dX}{m\sqrt{2E + 2kX}}, \quad m \neq 0, 2, 4. \quad (21)$$

Now fixing $(1-m)/m = n$, the above integral leads us to the following expression [32], that is,

$$t - t_0 = \int \frac{X^n dX}{m\sqrt{2E + 2kX}} = \frac{1}{m} \left(\frac{X^n \sqrt{2E + 2kX}}{2k(n + \frac{1}{2})} - \frac{2En}{2k(n + \frac{1}{2})} \int \frac{X^{n-1} dX}{\sqrt{2E + 2kX}} \right). \quad (22)$$

On the other hand fixing $(1-m)/m = -n$ in (21), we end up with the following expression [32]:

$$\begin{aligned} t - t_0 &= \int \frac{dX}{mX^n \sqrt{2E + 2kX}} \\ &= \frac{1}{m} \left(-\frac{\sqrt{2E + 2kX}}{2EX^{n-1}(n-1)} - \frac{k(n - \frac{3}{2})}{E(n-1)} \int \frac{dX}{X^{n-1} \sqrt{2E + 2kX}} \right). \end{aligned} \quad (23)$$

For integer values of n , the integrals (22) and (23) can be integrated explicitly by repeated use of these formulae. When n is a noninteger value, the final term in the integrals (22) and (23) can be integrated in terms of the logarithmic function or beta function [32] (since the value of $n-1$ in the final integral lies between 0 and 1). We also note here that for the special choices $n = 0$ or $-3/2$ (which corresponds to the case $\beta = \alpha^2/9$) and $n = -2/3$ or $-5/6$ (which corresponds to the case $\beta = -\alpha^2$) respectively in equations (22) and (23), one can get the respective solutions reported in the literature [1–5, 13–24].

3.3. Case 3: $\alpha^2 < 8\beta$

Finally, we focus our attention on the regime $\alpha^2 < 8\beta$. In this parametric regime, the Hamiltonian takes the form (vide equation (9))

$$H = \frac{1}{2} \log \left[x^4 \sec^2 \left(\frac{\omega}{4} x^2 p \right) \right] - \frac{\alpha}{4} x^2 p. \quad (24)$$

For the present case, we choose the canonical transformation in the form

$$x = \frac{U}{P}, \quad p = \frac{P^2}{2} \quad (25)$$

and transform the Hamiltonian to

$$H = \frac{1}{2} \log \left[\frac{U^4}{P^4} \sec^2 \left(\frac{\omega}{8} U^2 \right) \right] - \frac{\alpha}{8} U^2 \equiv E. \quad (26)$$

Even though one can also use the canonical transformation (12), we make use of the modified version (25) so as to have a separable form for H in the new coordinates.

When we make use of the canonical equations, $\dot{U} = -2/P$ and $\dot{P} = (1/4U)(\alpha - \omega \tan[(\omega/8)U^2])U^2 - 8$, equation (26) can be rewritten as

$$E = \frac{1}{2} \log \left[\frac{\dot{U}^4 U^4}{16} \sec^2 \left(\frac{\omega}{8} U^2 \right) \right] - \frac{\alpha}{8} U^2. \tag{27}$$

Introducing now the transformation $V = U^2/2$ in (27), we arrive at

$$E = \frac{1}{2} \log \left[\frac{\dot{V}^4}{16} \sec^2 \left(\frac{\omega}{4} V \right) \right] - \frac{\alpha}{4} V. \tag{28}$$

Integrating equation (28), we get

$$t - t_0 = \frac{\exp \left[-\frac{E}{2} \right]}{2} \int \sqrt{\sec \left(\frac{\omega}{4} V \right)} \exp \left[-\frac{\alpha}{8} V \right] dV, \tag{29}$$

where t_0 is the second constant of integration. By substituting $W = \exp[V]$ into (29), we obtain

$$t - t_0 = \frac{\exp \left[-\frac{E}{2} \right]}{\sqrt{2}} \int \frac{W^{q_1-1} dW}{\sqrt{1+W^{q_2}}} = \frac{\exp \left[-\frac{E}{2} \right] W^{q_1}}{\sqrt{2}q_1} F \left(\frac{1}{2}, \frac{q_1}{q_2}; 1 + \frac{q_1}{q_2}; -W^{q_2} \right), \tag{30}$$

where $q_1 = (-\alpha + i\omega)/8$, $q_2 = i\omega/2$ and $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function [31, 32].

4. Connection to two other nonlinear models

In this section we show that the MEE (1) is intimately connected with two other well-known nonlinear models, namely the equation of force-free Duffing oscillator type and two-dimensional Lotka–Volterra (LV) equation.

4.1. Equation of force-free Duffing oscillator type

The MEE equation (1) can be transformed to the following oscillator equation:

$$w'' + (\alpha w + \gamma)w' + \beta w^3 + \frac{\alpha\gamma}{3}w^2 + \frac{2\gamma^2}{9}w = 0, \quad \left(' = \frac{d}{d\tau} \right) \tag{31}$$

through the invertible point transformation $x = w \exp[(\gamma/3)\tau]$, $t = -(3/\gamma) \exp[-(\gamma/3)\tau]$, where γ is an arbitrary parameter. Equation (31) includes the force-free Duffing oscillator (in the specific case $\alpha = 0$) [23, 25, 26] and quadratic oscillator with $\beta = 0$ [23].

One can deduce the general solution of equation (31) from the general solution of MEE. For example, in the parametric choice $\beta = \alpha^2/8$ the general solution of equation (31) can be derived from (17) in the form

$$w(\tau) = \frac{8}{\alpha i} \operatorname{erf}^{-1}(\hat{z}) \exp \left[\frac{1}{2} \left(E - \frac{2}{3} \gamma \tau + 2[\operatorname{erf}^{-1}(\hat{z})]^2 \right) \right], \tag{32}$$

where now $\hat{z} = 2(\exp[E/2](\tau_0 - (3i/\gamma) \exp[-(\gamma/3)\tau]) / (\sqrt{\pi}))$ and E and τ_0 are arbitrary integration constants. Similarly, one can fix the general solution for equation (31) in the other parametric regimes also, that is, $\alpha^2 > 8\beta$ and $\alpha^2 < 8\beta$.

4.2. Two-dimensional Lotka–Volterra equation

Consider the two-dimensional Lotka–Volterra equation of the form

$$\dot{x} = x(a_1 + a_2x + a_3y), \quad \dot{y} = y(b_1 + b_2x + b_3y), \quad (33)$$

where a_i and b_i , $i = 1, 2, 3$, are six real parameters. Equation (33) models two species in competition in ecology and has been analysed for the past three decades or so in mathematical biology [27–29]. Interestingly, equation (33) can also be transformed to (31) as follows. Rewriting the first equation in (33) for the variable y and substituting it into the second equation in equation (33), we get the following second-order ODE for x , namely

$$\begin{aligned} \ddot{x} - \left(1 + \frac{b_3}{a_3}\right) \frac{\dot{x}^2}{x} + \left(\left(2a_2 \frac{b_3}{a_3} - a_2 - b_2\right)x + \left(2a_1 \frac{b_3}{a_3} - b_1\right)\right) \dot{x} + \left(b_2a_2 - \frac{b_3}{a_3}a_2^2\right)x^3 \\ + \left(a_2b_1 + b_2a_1 - 2a_1a_2 \frac{b_3}{a_3}\right)x^2 + \left(a_1b_1 - \frac{b_3}{a_3}a_1^2\right)x = 0. \end{aligned} \quad (34)$$

We choose the parameters in (34) in forms $b_3 = -a_3$ and $b_1 = a_1$ so that equation (34) can be brought to the form

$$\ddot{x} - ((3a_2 + b_2)x + 3a_1) \dot{x} + a_2(a_2 + b_2)x^3 + a_1(3a_2 + b_2)x^2 + 2a_1^2x = 0. \quad (35)$$

The associated LV equation takes the form

$$\dot{x} = x(a_1 + a_2x + a_3y), \quad \dot{y} = y(a_1 + b_2x - a_3y). \quad (36)$$

Now comparing equations (31) and (35), we obtain $\alpha = -(3a_2 + b_2)$, $\beta = a_2(a_2 + b_2)$ and $\gamma = -3a_1$. Choosing $a_2 = b_2$ ($\beta = \alpha^2/8$) the general solution for the LV equation (35) can be obtained from (32) and using this in the first equation in (36) we arrive at the general solution for the LV equation (36) in the form

$$\begin{aligned} x(t) &= \frac{2i}{a_2} \operatorname{erf}^{-1}(z) \exp \left[\frac{1}{2} \left(E + 2a_1t + 2[\operatorname{erf}^{-1}(z)]^2 \right) \right], \\ y(t) &= i \frac{\exp \left[\frac{1}{2} \left(E + 2a_1t + 2[\operatorname{erf}^{-1}(z)]^2 \right) \right]}{a_3 \operatorname{erf}^{-1}(z)}, \end{aligned} \quad (37)$$

where $z = (2 \exp[E/2](t_0 + i \exp[a_1t]/a_1))/(\sqrt{\pi})$ and E and t_0 are arbitrary integration constants. Finally, we mention that the general solution for equation (35) for other parametric choices can be obtained from equation (31) in a similar manner. Again (to our knowledge) the complete integrability of the LV system (36) in these regimes is new to the literature.

5. Conclusions

In this paper, we have shown that the MEE (1) is integrable either explicitly or by quadratures for any value of α and β . We have also obtained the time-independent Hamiltonians for equation (1). We have transformed the Hamiltonians into simpler forms, with appropriate canonical transformations, and deduced the general solution by direct integration so that our approach helps us to understand the dynamics of equation (1) in phase-space clearly. Further we have demonstrated that the complete integrability of equation (1) also helps one to understand the dynamics of two other nonlinear models, namely the generalized oscillator equation and the two-dimensional LV equation. As a consequence, the solutions which we have explored for equation (1) also provide solutions for these models as well.

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Appendix. Method of deriving integrals of motions

In the following, we briefly explain the generalized extended or modified Prele–Singer (PS) procedure for second-order ODEs [22, 23, 30] which is used to identify the integrals of motions (2)–(4).

To begin, we rewrite equation (1) in the form

$$\ddot{x} = -(\alpha x \dot{x} + \beta x^3) \equiv \phi(x, \dot{x}). \quad (\text{A.1})$$

Further we assume that the ODE (A.1) admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential becomes

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \quad (\text{A.2})$$

where each subscript denotes partial differentiation with respect to that variable. Rewriting equation (A.1) in the form $\phi dt - d\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x} dt - S(t, x, \dot{x}) dx$ to the latter, we obtain that on the solutions the 1-form

$$(\phi + S\dot{x}) dt - S dx - d\dot{x} = 0, \quad \phi = -(\alpha x \dot{x} + \beta x^3). \quad (\text{A.3})$$

Hence on the solutions the 1-forms (A.2) and (A.3) must be proportional. Multiplying (A.3) by the factor $R(t, x, \dot{x})$ which acts as the integrating factors for (A.3), we have on the solutions that

$$dI = R(\phi + S\dot{x}) dt - RS dx - R d\dot{x} = 0. \quad (\text{A.4})$$

Comparing equation (A.2) with (A.4), we have on the solutions the relations

$$I_t = R(\phi + \dot{x}S), \quad I_x = -RS, \quad I_{\dot{x}} = -R. \quad (\text{A.5})$$

Then the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between equations (A.5), provide us

$$S_t + \dot{x}S_x + \phi S_{\dot{x}} = -\phi_x + \phi_{\dot{x}}S + S^2, \quad (\text{A.6})$$

$$R_t + \dot{x}R_x + \phi R_{\dot{x}} = -(\phi_{\dot{x}} + S)R, \quad (\text{A.7})$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \quad (\text{A.8})$$

Solving equations (A.6)–(A.8), one can obtain expressions for S and R . It may be noted that any set of special solutions (S, R) is sufficient for our purpose. Once these forms are determined, the integral of motion $I(t, x, \dot{x})$ can be deduced from the relation

$$I = r_1 - r_2 - \int \left[R + \frac{d}{d\dot{x}}(r_1 - r_2) \right] d\dot{x}, \quad (\text{A.9})$$

where

$$r_1 = \int R(\phi + \dot{x}S) dt, \quad r_2 = \int \left(RS + \frac{d}{dx}r_1 \right) dx.$$

Equation (A.9) can be derived straightforwardly by integrating equation (A.5).

As our motivation is to explore the time-independent integral of motion for equation (A.1), we choose $I_t = 0$. In this case, one can easily fix the null form S from the first equation in (A.5) as

$$S = \frac{-\phi}{\dot{x}} = \frac{(\alpha x \dot{x} + \beta x^3)}{\dot{x}}. \quad (\text{A.10})$$

Substituting this form of S into (A.7), we get

$$\dot{x} R_x - (\alpha x \dot{x} + \beta x^3) R_{\dot{x}} = -\frac{\beta x^3}{\dot{x}} R. \quad (\text{A.11})$$

Equation (A.11) is a first-order linear partial differential equation with variable coefficients. As we noted earlier, any particular solution is sufficient to construct an integral of motion (along with the function S). To seek a particular solution for R , one can make a suitable ansatz instead of looking for the general solution. We assume R to be of the form

$$R = \frac{\dot{x}}{(A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r}, \quad (\text{A.12})$$

where A , B and C are functions of their arguments, and r is a constant which are all to be determined. We demand the above form of ansatz (A.12), which is very important to derive the Hamiltonian structure associated with the given equation, due to the following reason. To deduce the first integral I we assume a rational form for I , that is, $I = f(x, \dot{x})/g(x, \dot{x})$, where f and g are arbitrary functions of x and \dot{x} , respectively, and are independent of t , from which we get $I_x = (f_x g - f g_x)/g^2$ and $I_{\dot{x}} = (f_{\dot{x}} g - f g_{\dot{x}})/g^2$. From (A.5) one can see that $R = I_{\dot{x}} = (f_{\dot{x}} g - f g_{\dot{x}})/g^2$, $S = I_x/I_{\dot{x}} = (f_x g - f g_x)/(f_{\dot{x}} g - f g_{\dot{x}})$ and $RS = I_x$, so that the denominator of the function S should be the numerator of the function R . Since the denominator of S is \dot{x} (vide equation (A.10)), we fixed the numerator of R as \dot{x} . To seek a suitable function in the denominator initially, one can consider an arbitrary form $R = \dot{x}/h(x, \dot{x})$. However, it is difficult to proceed with this choice of h . So let us assume that $h(x, \dot{x})$ is a function which is polynomial in \dot{x} . To begin we consider the case where h is quadratic in \dot{x} , that is, $h = A(x) + B(x)\dot{x} + C(x)\dot{x}^2$, which is a generalized version of the form considered in [23], where only the linear form in \dot{x} was investigated (that is $C(x) = 0$). Since R is in a rational form, while taking differentiation or integration the form of the denominator remains same, but the power of the denominator decreases or increases by a unit order from that of the initial one. So instead of considering h to be of the form $h = A(x) + B(x)\dot{x} + C(x)\dot{x}^2$, one may consider a more general form $h = (A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r$, where r is a constant to be determined. The parameter r plays an important role, as we see below.

Substituting (A.12) into (A.11) and solving the resultant equations, we arrive at the relation

$$r[\dot{x}(A_x + B_x \dot{x} + C_x \dot{x}^2) - (\alpha x \dot{x} + \beta x^3)(B + 2C\dot{x})] = -\alpha x(A + B\dot{x} + C\dot{x}^2). \quad (\text{A.13})$$

Solving equation (A.13) we can fix the forms of A , B , C and r and substituting them into equation (A.12) we can get the integrating factor R . Doing so, we find

$$R = \begin{cases} \frac{\dot{x}}{(\dot{x} + \frac{(r-1)}{2r}\alpha x^2)^r}, & \alpha^2 \geq 8\beta \\ \frac{\dot{x}}{(2\dot{x}^2 + \alpha x^2 \dot{x} + \beta x^4)}, & \alpha^2 < 8\beta \\ \dot{x}, & \alpha = 0 \end{cases} \quad (\text{A.14})$$

where $r = (\alpha/4\beta)[\alpha \pm \sqrt{\alpha^2 - 8\beta}]$. One can easily check that the functions S and R given in (A.10) and (A.14), respectively, satisfy (A.8) also. Finally, substituting R and S into the form (A.9) for the integral we get the integrals of motion (2)–(4).

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